Works on an information geometrodynamical approach to chaos

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In this paper, I propose a theoretical information-geometric framework suitable to characterize chaotic dynamical behavior of arbitrary complex systems on curved statistical manifolds. Specifically, I present an information-geometric analogue of the Zurek-Paz quantum chaos criterion of linear entropy growth and an information-geometric characterization of regular and chaotic quantum energy level statistics.

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I. INTRODUCTION

The study of complexity [1] has created a new set of ideas on how very simple systems may give rise to very complex behaviors. Moreover, in many cases, the "laws of complexity" have been found to hold universally, caring not at all for the details of the system's constituents. Chaotic behavior is a particular case of complex behavior and it will be the object of the present work.

In this paper I make use of the so-called Entropic Dynamics (ED) [2]. ED is a theoretical framework that arises from the combination of inductive inference (Maximum Entropy Methods (ME), [3]) and Information Geometry (IG) [4]. The most intriguing question being pursued in ED stems from the possibility of deriving dynamics from purely entropic arguments. This is clearly valuable in circumstances where microscopic dynamics may be too far removed from the phenomena of interest, such as in complex biological or ecological systems, or where it may just be unknown or perhaps even nonexistent, as in economics. It has already been shown that entropic arguments do account for a substantial part of the formalism of quantum mechanics, a theory that is presumably fundamental [5]. Perhaps the fundamental theories of physics are not so fundamental; they may just be consistent, objective ways to manipulate information. Following this line of thought, I extend the applicability of ED to temporally-complex (chaotic) dynamical systems on curved statistical manifolds and identify relevant measures of chaoticity of such an information geometrodynamical approach to chaos (IGAC).

The layout of the paper is as follows. In the next Section, I give an introduction to the main features of our IGAC. In Section III, I apply my theoretical construct to three complex systems. First, I study the chaotic behavior of an ED Gaussian model describing an arbitrary system of l degrees of freedom and show that the hyperbolicity of the non-maximally symmetric $2l$-dimensional statistical manifold $\mathcal{M}_s$ underlying such ED Gaussian model leads to linear information geometrodynamical entropy (IGE) growth and to exponential divergence of the Jacobi vector field intensity. An information-geometric analogue of the Zurek-Paz quantum chaos criterion of linear entropy growth and an information-geometric characterization of regular and chaotic quantum energy level statistics are presented.

I emphasize that I have omitted technical details that will appear elsewhere. However, some applications of my IGAC to low dimensional chaotic systems can be found in my previous articles [6, 7, 8, 9]. Finally, in Section IV I present my conclusions and suggest further research directions.

II. THE INFORMATION GEOMETRODYNAMICAL APPROACH TO CHAOS: GENERAL FORMALISM

The IGAC is an application of ED to complex systems of arbitrary nature. ED is a form of information-constrained dynamics built on curved statistical manifolds $\mathcal{M}_S$ where elements of the manifold are probability distributions $\{P(X|\Theta)\}$ that are in a one-to-one relation with a suitable set of macroscopic statistical variables $\{\Theta\}$ that provide a convenient parametrization of points on $\mathcal{M}_S$. The set $\{\Theta\}$ is called the *parameter space* $\mathcal{D}_\Theta$ of the system.

In what follows, I schematically outline the main points underlying the construction of an arbitrary form of entropic dynamics. First, the microstates of the system under investigation must be defined. For the sake of simplicity, I

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assume the system is characterized by an l-dimensional *microspace* with microstates $\{x_k\}$ where $k = 1,..., l$. These microstates can be interacting, non-interacting, or I may have no relevant information concerning their microscopic dynamics. Indeed, the main goal of an ED model is that of inferring "macroscopic predictions" in the absence of detailed knowledge of the microscopic nature of arbitrary complex systems. Once the microstates have been defined, I have to select the relevant information about such microstates. In other words, I have to select the *macrospace* of the system. For the sake of the argument, I assume that our microstates are Gaussian-distributed. They are defined by $2l$-information constraints, for example their expectation values μ_k and variances σ_k.

$$\langle x_k \rangle \equiv \mu_k \text{ and } \left(\left\langle (x_k - \langle x_k \rangle)^2 \right\rangle\right)^{\frac{1}{2}} \equiv \sigma_k. \tag{1}$$

In addition to information constraints, each Gaussian distribution $p_k(x_k|\mu_k, \sigma_k)$ of each microstate x_k must satisfy the usual normalization conditions,

$$\int_{-\infty}^{+\infty} dx_k p_k(x_k|\mu_k, \sigma_k) = 1 \tag{2}$$

where

$$p_k(x_k|\mu_k, \sigma_k) = \left(2\pi\sigma_k^2\right)^{-\frac{1}{2}} \exp\left(-\frac{(x_k - \mu_k)^2}{2\sigma_k^2}\right). \tag{3}$$

Once the microstates have been defined and the relevant (linear or nonlinear) information constraints selected, I am left with a set of probability distributions $p(X|\Theta) = \prod_{k=1}^{l} p_k(x_k|\mu_k, \sigma_k)$ encoding the relevant available information about the system where X is the l-dimensional microscopic vector with components $(x_1,...,x_l)$ and Θ is the $2l$-dimensional macroscopic vector with coordinates $(\mu_1,..., \mu_l; \sigma_1,..., \sigma_l)$. The set $\{\Theta\}$ define the $2l$-dimensional space of macrostates of the system, the statistical manifold $\mathcal{M}_S$. A measure of distinguishability among macrostates is obtained by assigning a probability distribution $P(X|\Theta) \ni \mathcal{M}_S$ to each macrostate Θ . Assignment of a probability distribution to each state endows $\mathcal{M}_S$ with a metric structure. Specifically, the Fisher-Rao information metric $g_{\mu\nu}(\Theta)$ [4],

$$g_{\mu\nu}(\Theta) = \int dX p(X|\Theta)\, \partial_\mu \log p(X|\Theta)\, \partial_\nu \log p(X|\Theta), \tag{4}$$

with μ,$\nu = 1,..., 2l$ and $\partial_\mu = \frac{\partial}{\partial \Theta^\mu}$, defines a measure of distinguishability among macrostates on $\mathcal{M}_S$. The statistical manifold $\mathcal{M}_S$,

$$\mathcal{M}_S = \left\{p(X|\Theta) = \prod_{k=1}^{l} p_k(x_k|\mu_k, \sigma_k)\right\}, \tag{5}$$

is defined as the set of probabilities $\{p(X|\Theta)\}$ described above where $X \in \mathbb{R}^{3N}$, $\Theta \in \mathcal{D}_\Theta = [\mathcal{I}_\mu \times \mathcal{I}_\sigma]^{3N}$. The parameter space $\mathcal{D}_\Theta$ (homeomorphic to $\mathcal{M}_S$) is the direct product of the parameter subspaces $\mathcal{I}_\mu$ and $\mathcal{I}_\sigma$, where (unless specified otherwise) $\mathcal{I}_\mu = (-\infty, +\infty)_\mu$ and $\mathcal{I}_\sigma = (0, +\infty)_\sigma$. Once $\mathcal{M}_S$ and $\mathcal{D}_\Theta$ are defined, the ED formalism provides the tools to explore dynamics driven on $\mathcal{M}_S$ by entropic arguments. Specifically, given a known initial macrostate $\Theta^{(\text{initial})}$ (probability distribution), and that the system evolves to a final known macrostate $\Theta^{(\text{final})}$, the possible trajectories of the system are examined in the ED approach using ME methods.

I emphasize ED can be derived from a standard principle of least action (of Jacobi type). The geodesic equations for the macrovariables of the Gaussian ED model are given by *nonlinear* second order coupled ordinary differential equations,

$$\frac{d^2\Theta^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{d\Theta^\nu}{d\tau} \frac{d\Theta^\rho}{d\tau} = 0. \tag{6}$$

The geodesic equations in (6) describe a *reversible* dynamics whose solution is the trajectory between an initial $\Theta^{(\text{initial})}$ and a final macrostate $\Theta^{(\text{final})}$. The trajectory can be equally well traversed in both directions. Given the Fisher-Rao information metric, I can apply standard methods of Riemannian differential geometry to study the information-geometric structure of the manifold $\mathcal{M}_S$ underlying the entropic dynamics. Connection coefficients $\Gamma^\rho_{\mu\nu}$, Ricci tensor

$R_{\mu\nu}$, Riemannian curvature tensor $R_{\mu\nu\rho\sigma}$, sectional curvatures $\mathcal{K}_{\mathcal{M}_S}$, scalar curvature $\mathcal{R}_{\mathcal{M}_S}$, Weyl anisotropy tensor $W_{\mu\nu\rho\sigma}$, Killing fields ξ^{μ} and Jacobi fields J^{μ} can be calculated in the usual way.

To characterize the chaotic behavior of complex entropic dynamical systems, I are mainly concerned with the signs of the scalar and sectional curvatures of $\mathcal{M}_S$, the asymptotic behavior of Jacobi fields J^{μ} on $\mathcal{M}_S$, the existence of Killing vectors ξ^{μ} (or existence of a non-vanishing Weyl anisotropy tensor, the anisotropy of the manifold underlying system dynamics plays a significant role in the mechanism of instability) and the asymptotic behavior of the information-geometrodynamical entropy (IGE) $\mathcal{S}_{\mathcal{M}_S}$ (see (9)). It is crucial to observe that true chaos is identified by the occurrence of two features: 1) strong dependence on initial conditions and exponential divergence of the Jacobi vector field intensity, i.e., *stretching* of dynamical trajectories; 2) compactness of the configuration space manifold, i.e., *folding* of dynamical trajectories. The negativity of the Ricci scalar $\mathcal{R}_{\mathcal{M}_S}$,

$$\mathcal{R}_{\mathcal{M}_S} = R_{\mu\nu\rho\sigma} g^{\mu\rho} g^{\nu\sigma} = \sum_{\rho\neq\sigma} \mathcal{K}_{\mathcal{M}_S}\left(e_{\rho},\, e_{\sigma}\right), \tag{7}$$

implies the existence of expanding directions in the configuration space manifold $\mathcal{M}_s$. Indeed, since $\mathcal{R}_{\mathcal{M}_S}$ is the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements $\left\{e_{\rho} = \partial_{\Theta_{\rho}}\right\}$, the negativity of the Ricci scalar is only a *sufficient* (not necessary) condition for local instability of geodesic flow. For this reason, the negativity of the scalar provides a *strong* criterion of local instability. Scenarios may arise where negative sectional curvatures are present, but the positive ones could prevail in the sum so that the Ricci scalar is non-negative despite the instability in the flow in those directions. Consequently, the signs of $\mathcal{K}_{\mathcal{M}_S}$ are of primary significance for the proper characterization of chaos.

A powerful mathematical tool to investigate the stability or instability of a geodesic flow is the Jacobi-Levi-Civita equation (JLC equation) for geodesic spread,

$$\frac{D^2 J^{\mu}}{D\tau^2} + R^{\mu}_{\nu\rho\sigma} \frac{\partial\Theta^{\nu}}{\partial\tau} J^{\rho} \frac{\partial\Theta^{\sigma}}{\partial\tau} = 0. \tag{8}$$

The JLC-equation covariantly describes how nearby geodesics locally scatter and relates the stability or instability of a geodesic flow with curvature properties of the ambient manifold. Finally, the asymptotic regime of diffusive evolution describing the possible exponential increase of average volume elements on $\mathcal{M}_s$ provides another useful indicator of dynamical chaoticity. The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the statistical manifold. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos where the entropy (von Neumann) increases linearly at a rate determined by the Lyapunov exponents. The linear increase of entropy as a quantum chaos criterion was introduced by Zurek and Paz [10]. In my information-geometric approach a relevant quantity that can be useful to study the degree of instability characterizing ED models is the information geometrodynamical entropy (IGE) defined as [7],

$$\mathcal{S}_{\mathcal{M}_s}(\tau) \overset{\text{def}}{=} \lim_{\tau\to\infty} \log \mathcal{V}_{\mathcal{M}_s} \text{ with } \mathcal{V}_{\mathcal{M}_s}(\tau) = \frac{1}{\tau}\int_0^{\tau} d\tau' \left(\int_{\mathcal{M}_s} \sqrt{g}\, d^{2l}\Theta\right) \tag{9}$$

and $g = \left|\det\left(g_{\mu\nu}\right)\right|$. IGE is the asymptotic limit of the natural logarithm of the statistical weight defined on $\mathcal{M}_s$ and represents a measure of temporal complexity of chaotic dynamical systems whose dynamics is underlined by a curved statistical manifold. In conventional approaches to chaos, the notion of entropy is introduced, in both classical and quantum physics, as the missing information about the systems fine-grained state [11]. For a classical system, suppose that the phase space is partitioned into very fine-grained cells of uniform volume Δv, labelled by an index j. If one does not know which cell the system occupies, one assigns probabilities p_j to the various cells; equivalently, in the limit of infinitesimal cells, one can use a phase-space density $\rho\left(X_j\right) = \frac{p_j}{\Delta v}$. Then, in a classical chaotic evolution, the asymptotic expression of the information needed to characterize a particular coarse-grained trajectory out to time τ is given by the Shannon information entropy (measured in bits),

$$\begin{aligned} \mathcal{S}^{(\text{chaotic})}_{\text{classical}} &= -\int dX \rho\left(X\right) \log_2\left(\rho\left(X\right)\Delta v\right) \\ &= -\sum_j p_j \log_2 p_j \sim \mathcal{K}\tau. \end{aligned} \tag{10}$$

where $\rho\left(X\right)$ is the phase-space density and $p_j = \frac{v_j}{\Delta v}$ is the probability for the corresponding coarse-grained trajectory. $\mathcal{S}^{(\text{chaotic})}_{\text{classical}}$ is the missing information about which fine-grained cell the system occupies. The quantity $\mathcal{K}$ represents the linear rate of information increase and it is called the Kolmogorov-Sinai entropy (or metric entropy) ($\mathcal{K}$ is the sum of positive Lyapunov exponents, $\mathcal{K} = \sum_j \lambda_j$). $\mathcal{K}$ quantifies the degree of classical chaos.

III. THE INFORMATION GEOMETRODYNAMICAL APPROACH TO CHAOS: APPLICATIONS

In this Section I present three applications of the IGAC. First, I study the chaotic behavior of an ED Gaussian model describing an arbitrary system of l degrees of freedom and show that the hyperbolicity of the non-maximally symmetric $2l$-dimensional statistical manifold $\mathcal{M}_s$ underlying such ED Gaussian model leads to linear information geometrodynamical entropy (IGE) growth and to exponential divergence of the Jacobi vector field intensity. I also present an information-geometric analogue of the Zurek-Paz quantum chaos criterion of linear entropy growth. This analogy is presented by studying the information geometrodynamics of ensemble of random frequency macroscopic inverted harmonic oscillators. Finally, I apply the IGAC to study the entropic dynamics on curved statistical manifolds induced by classical probability distributions of common use in the study of regular and chaotic quantum energy level statistics. In doing so, I suggest an information-geometric characterization of regular and chaotic quantum energy level statistics.

As I said in the Introduction, I have omitted technical details that will appear elsewhere. However, my previous works (especially ([7])) may be very useful references in order to clarify the following applications.

A. Chaotic behavior of an entropic dynamical Gaussian model

As a first example, I apply my IGAC to study the dynamics of a system with l degrees of freedom, each one described by two pieces of relevant information, its mean expected value and its variance (Gaussian statistical macrostates). The line element $ds^2 = g_{\mu\nu}(\Theta)\, d\Theta^{\mu} d\Theta^{\nu}$ on $\mathcal{M}_s$ is defined by,

$$ds^2 = \sum_{k=1}^{l} \left(\frac{1}{\sigma_k^2} d\mu_k^2 + \frac{2}{\sigma_k^2} d\sigma_k^2 \right), \text{ with } \mu,\, \nu = 1,...,\, 2l. \tag{11}$$

This leads to consider an ED model on a non-maximally symmetric $2l$-dimensional statistical manifold $\mathcal{M}_s$. It is shown that $\mathcal{M}_s$ possesses a constant negative Ricci curvature that is proportional to the number of degrees of freedom of the system, $R_{\mathcal{M}_s} = -l$. It is shown that the system explores statistical volume elements on $\mathcal{M}_s$ at an exponential rate. The information geometrodynamical entropy $\mathcal{S}_{\mathcal{M}_s}$ increases linearly in time (statistical evolution parameter) and is moreover, proportional to the number of degrees of freedom of the system, $\mathcal{S}_{\mathcal{M}_s} \overset{\tau\to\infty}{\sim} l\lambda\tau$. The parameter λ characterizes the family of probability distributions on $\mathcal{M}_s$. The asymptotic linear information-geometrodynamical entropy growth may be considered the information-geometric analogue of the von Neumann entropy growth introduced by Zurek-Paz, a *quantum* feature of chaos. The geodesics on $\mathcal{M}_s$ are hyperbolic trajectories. Using the Jacobi-Levi-Civita (JLC) equation for geodesic spread, I show that the Jacobi vector field intensity $J_{\mathcal{M}_s}$ diverges exponentially and is proportional to the number of degrees of freedom of the system, $J_{\mathcal{M}_s} \overset{\tau\to\infty}{\sim} l\exp(\lambda\tau)$. The exponential divergence of the Jacobi vector field intensity $J_{\mathcal{M}_s}$ is a *classical* feature of chaos. Therefore, we conclude that

$$\mathcal{R}_{\mathcal{M}_s} = -l,\; J_{\mathcal{M}_s} \overset{\tau\to\infty}{\sim} l\exp(\lambda\tau),\; \mathcal{S}_{\mathcal{M}_s} \overset{\tau\to\infty}{\sim} l\lambda\tau. \tag{12}$$

Thus, $\mathcal{R}_{\mathcal{M}_s}$, $\mathcal{S}_{\mathcal{M}_s}$ and $J_{\mathcal{M}_s}$ behave as proper indicators of chaoticity and are proportional to the number of Gaussian-distributed microstates of the system. This proportionality, even though proven in a very special case, leads to conclude there may be a substantial link among these information-geometric indicators of chaoticity.

B. Ensemble of random frequency macroscopic inverted harmonic oscillators

In our second example, I employ ED and "Newtonian Entropic Dynamics" (NED) [9]. In this NED, we explore the possibility of using well established principles of inference to derive Newtonian dynamics from relevant prior information codified into an appropriate statistical manifold. The basic assumption is that there is an irreducible uncertainty in the location of particles so that the state of a particle is defined by a probability distribution. The corresponding configuration space is a statistical manifold $\mathcal{M}_s$ the geometry of which is defined by the Fisher-Rao information metric. The trajectory follows from a principle of inference, the method of Maximum Entropy. There is no need for additional "physical" postulates such as an action principle or equation of motion, nor for the concept of mass, momentum and of phase space, not even the notion of time. The resulting "entropic" dynamics reproduces Newton's mechanics for any number of particles interacting among themselves and with external fields. Both the mass of the particles and their interactions are explained as a consequence of the underlying statistical manifold.

In my special application, I consider a manifold with a line element $ds^2 = g_{\mu\nu}(\Theta)\, d\Theta^\mu d\Theta^\nu$ (with μ, $\nu = 1,..., l$) given by,

$$ds^2 = [1 - \Phi(\Theta)]\, \delta_{\mu\nu}(\Theta)\, d\Theta^\mu d\Theta^\nu,\ \Phi(\Theta) = \sum_{k=1}^{l} u_k(\theta_k) \tag{13}$$

where

$$u_k(\theta_k) = -\frac{1}{2}\omega_k^2\theta_k^2,\ \theta_k = \theta_k(s)\,. \tag{14}$$

The geodesic equations for the macrovariables $\theta_k(s)$ are strongly *nonlinear* and their integration is not trivial. However, upon a suitable change of the affine parameter s used in the geodesic equations, I may simplify the differential equations for the macroscopic variables parametrizing points on the manifold $\mathcal{M}_s$ with metric tensor $g_{\mu\nu}$. Recalling that the notion of chaos is observer-dependent and upon changing the affine parameter from s to τ in such a way that $ds^2 = 2(1-\Phi)^2 d\tau^2$, I obtain new geodesic equations describing a set of macroscopic inverted harmonic oscillators (IHOs). In order to ensure the compactification of the parameter space of the system (and therefore $\mathcal{M}_s$ itself), we choose a Gaussian distributed frequency spectrum for the IHOs. Thus, with this choice of frequency spectrum, the folding mechanism required for true chaos is restored in a statistical (averaging over ω and τ) sense. Upon integrating these differential equations, I obtain the expression for the asymptotic behavior of the IGE $\mathcal{S}_{\mathcal{M}_s}$, namely

$$\mathcal{S}_{\mathcal{M}_s}(\tau) \overset{\tau\to\infty}{\sim} \Lambda\tau,\ \Lambda = \sum_{i=1}^{l}\omega_i. \tag{15}$$

This result may be considered the information-geometric analogue of the Zurek-Paz model used to investigate the implications of decoherence for quantum chaos. In their work, Zurek and Paz considered a chaotic system, a single unstable harmonic oscillator characterized by a potential $V(x) = -\frac{\Omega^2 x^2}{2}$ (Ω is the Lyapunov exponent), coupled to an external environment. In the *reversible classical limit* [12], the von Neumann entropy of such a system increases linearly at a rate determined by the Lyapunov exponent,

$$\mathcal{S}_{\text{quantum}}^{(\text{chaotic})}(\tau) \overset{\tau\to\infty}{\sim} \Omega\tau, \tag{16}$$

with Ω playing the role of the Lyapunov exponent.

C. Information geometrodynamics of regular and chaotic quantum spin chains

In my final example, I use my IGAC to study the entropic dynamics on curved statistical manifolds induced by classical probability distributions of common use in the study of regular and chaotic quantum energy level statistics. Recall that the theory of quantum chaos (quantum mechanics of systems whose classical dynamics are chaotic) is not primarily related to few-body physics. Indeed, in real physical systems such as many-electron atoms and heavy nuclei, the origin of complex behavior is the very strong interaction among many particles. To deal with such systems, a famous statistical approach has been developed which is based upon the Random Matrix Theory (RMT). The main idea of this approach is to neglect the detailed description of the motion and to treat these systems statistically bearing in mind that the interaction among particles is so complex and strong that generic properties are expected to emerge. Once again, this is exactly the philosophy underlining the ED approach to complex dynamics. It is known [13] that integrable and chaotic quantum antiferromagnetic Ising chains are characterized by asymptotic logarithmic and linear growths of their operator space entanglement entropies, respectively. In this last example, I consider the information-geometrodynamics of a Poisson distribution coupled to an Exponential bath (spin chain in a *transverse* magnetic field, regular case) and that of a Wigner-Dyson distribution coupled to a Gaussian bath (spin chain in a *tilted* magnetic field, chaotic case). Remarkably, I show that in the former case the IGE exhibits asymptotic logarithmic growth while in the latter case the IGE exhibits asymptotic linear growth. In the regular case, the line element $ds^2_{\text{integrable}} = ds^2_{\text{Poisson}} + ds^2_{\text{Exponential}}$ on the statistical manifold $\mathcal{M}_s$ is given by

$$ds^2_{\text{integrable}} = \frac{1}{\mu_A^2}d\mu_A^2 + \frac{1}{\mu_B^2}d\mu_B^2 \tag{17}$$

where the macrovariable μ_A is the average spacing of the energy levels and μ_B is the average intensity of the magnetic energy arising from the interaction of the *transverse* magnetic field with the spin $\frac{1}{2}$ particle magnetic moment. In

such a case, I show that the asymptotic behavior of $\mathcal{S}_{\mathcal{M}_s}^{(\text{integrable})}$ is sub-linear in τ (logarithmic IGE growth),

$$\mathcal{S}_{\mathcal{M}_s}^{(\text{integrable})}(\tau) \overset{\tau\to\infty}{\sim} \log\tau. \tag{18}$$

Finally, in the chaotic case, the line element $ds^2_{\text{chaotic}} = ds^2_{\text{Wigner-Dyson}} + ds^2_{\text{Gaussian}}$ on the statistical manifold $\mathcal{M}_s$ is given by

$$ds^2_{\text{chaotic}} = \frac{4}{\mu_A'^2} d\mu_A'^2 + \frac{1}{\sigma_B'^2} d\mu_B'^2 + \frac{2}{\sigma_B'^2} d\sigma_B'^2 \tag{19}$$

where the (nonvanishing) macrovariable μ_A' is the average spacing of the energy levels, μ_B' and σ_B' are the average intensity and variance, respectively of the magnetic energy arising from the interaction of the *tilted* magnetic field with the spin $\frac{1}{2}$ particle magnetic moment. In this case, I show that asymptotic behavior of $\mathcal{S}_{\mathcal{M}_s}^{(\text{chaotic})}$ is linear in τ (linear IGE growth),

$$\mathcal{S}_{\mathcal{M}_s}^{(\text{chaotic})}(\tau) \overset{\tau\to\infty}{\sim} \tau. \tag{20}$$

The equations for $\mathcal{S}_{\mathcal{M}_s}^{(\text{integrable})}$ and $\mathcal{S}_{\mathcal{M}_s}^{(\text{chaotic})}$ are the information-geometric analogue of the entanglement entropies defined in standard quantum information theory in the regular and chaotic cases, respectively. In addition, I emphasize that the statistical volume element $\mathcal{V}_{\mathcal{M}_s}(\tau)$ (see (9)) may play a similar role as the computational complexity in conventional quantum information theory. These results warrant deeper analysis to be fully understood.

IV. CONCLUSION

In this paper I proposed a theoretical information-geometric framework suitable to characterize chaotic dynamical behavior of arbitrary complex systems on curved statistical manifolds. Specifically, an information-geometric analogue of the Zurek-Paz quantum chaos criterion of linear entropy growth and an information-geometric characterization of regular and chaotic quantum energy level statistics was presented. I hope that my work convincingly shows that this information-geometric approach may be useful in providing a unifying criterion of chaos of both classical and quantum varieties, thus deserving further research and developments.

The descriptions of a classical chaotic system of arbitrary interacting degrees of freedom, deviations from Gaussianity and chaoticity arising from fluctuations of positively curved statistical manifolds are currently under investigation. I am also investigating the possibility to extend the IGAC to quantum Hilbert spaces constructed from classical curved statistical manifolds and I am considering the information-geometric macroscopic versions of the Henon-Heiles and Fermi-Pasta-Ulam β-models to study chaotic geodesic flows on statistical manifolds.

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